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Nonlinear equations based on jointly homogeneous mappings

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Abstract

In this paper, we consider a nonlinear equation based on two-variable monotone and jointly homogeneous functions on an open convex normal cone

$$x = g(h(a, x), k(b, x))$$

and show that it has a unique positive solution and furthermore the solution map is again monotone and jointly homogeneous. We apply our results to nonlinear matrix equations involving the geometric mean of positive definite matrices, which include some known nonlinear matrix equations.

Keywords: Normal cone; Thompson's part metric; Geometric mean; Jointly homogeneous map; Positive solution

1. Introduction

In [2], Ando et al. have found a remarkable property of the geometric mean of positive definite matrices, namely the *limited medial property*

$$M = A\#B = C\#D \text{ implies } M = (A\#C)\#(B\#D) = (A\#D)\#(B\#C),$$

where $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ denotes the geometric mean of A and B. The geometric meaning of the limited medial property is straightforward after viewing $A\#B$ as the unique midpoint (geodesic middle) of A and B for the $\mathrm{GL}(n, \mathbb{C})$-invariant Riemannian metric $\delta(A, B) = \|\log A^{-1/2}BA^{-1/2}\|_2$ (see [4,10,11]) and comparing with Euclidean geometry.

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Let $X = A\#B$. Then $X = (A\#B)\#(A\#B) = (A\#(A\#B))\#(B\#(A\#B))$ by the limited medial property and hence $X = A\#B$ is a positive definite solution of $X = (A\#X)\#(B\#X)$. Conversely, if X is a positive definite solution of $X = (A\#X)\#(B\#X)$, then by invariance under congruence transformation we have $I = (X^{-1/2}AX^{-1/2})^{1/2}\#(X^{-1/2}BX^{-1/2})^{1/2}$ or $X^{1/2}A^{-1}X^{1/2} = X^{-1/2}BX^{-1/2}$, where I is the identity matrix. This implies that X is a solution of the Riccati equation $XA^{-1}X = B$ and therefore $X = A\#B$. However, it is non-trivial to see that the (twisted) matrix equation $X = (A\#MXM^*)\#(B\#NXN^*)$ for fixed non-singular matrices M and N has a unique positive definite solution and also to analysis the solution map. As a special case if $M = N$ and $A = B = I$. then the equation $X = I\#MXM^* = (MXM^*)^{1/2}$ or $X^2 = MXM^*$ is studied by Bushell [5,6]. See also [8,9,15].

In this paper, we consider that for non-singular matrices M and N, the following general nonlinear matrix equation:

$$X = (A\#_\beta MXM^*)\#_\alpha(B\#_\gamma NXN^*), \quad \alpha, \beta, \gamma \in (0, 1),$$

where $A\#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$ is the α-weighted geometric mean of positive definite A and B. We prove that the equation has a unique positive definite solution and the solution map denoted by $\omega(A, B)$ varying over A and B is monotone and $\frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}$-jointly homogeneous. Here a map $f(X, Y)$ is μ-jointly homogeneous means that $f(sX, tY) = s^{1-\mu}t^\mu f(X, Y)$. We provide a proof in general context of open convex normal cones of Banach spaces that includes any finite dimensional open convex cones and the convex cone of positive elements of a C^*-algebra. A method constructing a sequence of monotone and α-jointly homogeneous maps from given such a function is considered. Furthermore, it is shown that the solution map varying over A of $X = \omega(A, X)$ is an order preserving homogeneous map of degree 1.

2. Thompson's part metric

Let V be a real Banach space and let Ω henceforth denote a non-empty open convex cone of $V : t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω. We consider the partial order on V defined by

$$x \leqslant y \quad \text{if and only if } y - x \in \overline{\Omega}.$$

We further assume that Ω is a *normal* cone: there exists a constant K with $\|x\| \leqslant K\|y\|$ for all $x, y \in \Omega$ with $x \leqslant y$. Any member a of Ω is an order unit for the ordered space $(V, \leqslant)$, and hence $|x|_a := \inf\{\lambda > 0 : -\lambda a \leqslant x \leqslant \lambda a\}$ defines a norm. By Proposition 1.1 in [16], for a normal cone $\overline{\Omega}$, the order unit norm $|\cdot|_a$ is equivalent to $\|\cdot\|$.

Thompson [17] (cf. [15,16]) has proved that Ω is a complete metric space with respect to the *Thompson part metric* defined by

$$d(x, y) = \max\{\log M(x/y), \log M(y/x)\},$$

where $M(x/y) := \inf\{\lambda > 0 : x \leqslant \lambda y\} = |x|_y$. Furthermore, the topology induced by the Thompson metric agrees with the relative Banach space topology. If Ω is the open convex cone of positive definite operators on a Hilbert space, then $d(A, B) = \|\log A^{-1/2}BA^{-1/2}\|$, where $\|\cdot\|$ denotes the spectral norm [15].

3. Jointly homogeneous mappings

Definition 3.1. A map $f : \Omega \to \Omega$ is called *homogeneous of degree r* if $f(tx) = t^r f(x)$ for any $t > 0$ and $x \in \Omega$.

Proposition 3.2 (cf. [15,16]). *Let $f : \Omega \to \Omega$ be an order preserving homogeneous map of degree $r \geqslant 0$. Then for any $x, y \in \Omega$*

$$d(f(x), f(y)) \leqslant rd(x, y).$$

In particular if $r \in [0, 1)$, then f is a strict contraction and has a unique fixed point in Ω.

Proof. The order preserving and homogeneous properties imply that

$$\{\lambda^r > 0 : x \leqslant \lambda y\} \subseteq \{\beta > 0 : f(x) \leqslant \beta f(y)\}$$

for all $x, y \in \Omega$. Thus, $M(x/y)^r \geqslant M(f(x)/f(y))$ and therefore $d(f(x), f(y)) \leqslant rd(x, y)$ for all $x, y \in \Omega$. If $0 \leqslant r < 1$, then f is a strict contraction for the Thompson metric and by completeness of the metric and Banach fixed point theorem, it has a unique fixed point in Ω. □

Definition 3.3. A mapping $g : \Omega^2 \to \Omega$ is called *monotone* if $g(a_1, a_2) \leqslant g(b_1, b_2)$ for $a_i \leqslant b_i$ for all $i = 1, 2$. Also, g is *μ-jointly homogeneous* if

$$g(sa_1, ta_2) = s^{1-\mu} t^{\mu} g(a_1, a_2)$$

for all $s, t > 0$.

Theorem 3.4. *Let $g, h, k : \Omega^2 \to \Omega$ be monotone and α, β, γ-jointly homogeneous maps, respectively. Assume that*

$$0 \leqslant \beta(1-\alpha) + \gamma\alpha < 1. \tag{3.1}$$

Then for $a, b \in \Omega$, the equation

$$x = g(h(a, x), k(b, x)) \tag{3.2}$$

has a unique solution in Ω, denoted by $\omega(a, b)$. Furthermore, the solution map $\omega : \Omega^2 \to \Omega$ is monotone and $\frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}$-jointly homogeneous.

Proof. Define $f : \Omega \to \Omega$ by $f(x) = g(h(a, x), k(b, x))$. If $x \leqslant y$ then $h(a, x) \leqslant h(a, y)$, $k(b, x) \leqslant k(b, y)$, and therefore

$$g(h(a, x), k(b, x)) \leqslant g(h(a, y), k(b, y)).$$

That is, f is order preserving. Let $t > 0$. Then $h(a, tx) = t^{\beta} h(a, x)$, $k(b, tx) = t^{\gamma} k(b, x)$ and hence

$$\begin{aligned} f(tx) &= g(h(a, tx), k(b, tx)) = g(t^{\beta} h(a, x), t^{\gamma} k(b, x)) \\ &= t^{\beta(1-\alpha)} t^{\gamma\alpha} g(h(a, x), k(b, x)) = t^{\beta(1-\alpha)+\gamma\alpha} f(x). \end{aligned}$$

Since $0 \leqslant \beta(1-\alpha) + \gamma\alpha < 1$, f is a strict contraction by Proposition 3.2 and hence it has a unique fixed point, which coincides with the unique positive solution of (3.2).

Next, we will show that ω is monotone. Let $a' \leqslant a$ and $b' \leqslant b$. We have seen that the function $f(x) := g(h(a, x), k(b, x))$ is a monotone and strict contraction for the Thompson metric. By Banach fixed point theorem, $\omega(a, b) = \lim_{n\to\infty} f^n(x)$ for any $x \in \Omega$. Setting $x_0 := \omega(a', b')$, we have

$$x_0 = g(h(a', x_0), k(b', x_0)) \leqslant g(h(a, x_0), k(b, x_0)) = f(x_0)$$

and by order preserving of f

$$x_0 \leqslant f(x_0) \leqslant f^2(x_0) \leqslant \cdots \leqslant f^n(x_0) \to \omega(a, b).$$

This implies that $\omega(a', b') = x_0 \leqslant \omega(a, b)$.

Finally, we will show that the solution map $\omega : \Omega^2 \to \Omega$ is $\mu := \frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}$-jointly homogeneous. To see this, we first note that

$$1 - \mu = (1 - \beta)(1 - \alpha) + (1 - \mu)\beta(1 - \alpha) + (1 - \mu)\alpha\gamma, \tag{3.3}$$

$$\mu = \mu\beta(1 - \alpha) + (1 - \gamma)\alpha + \mu\alpha\gamma. \tag{3.4}$$

Let $a, b \in \Omega$ and let $s, t > 0$. Set $x = \omega(a, b)$. Then the equation $\omega(sa, tb) = s^{1-\mu}t^{\mu}x$ is equivalent to $g(h(sa, s^{1-\mu}t^{\mu}x), k(tb, s^{1-\mu}t^{\mu}x)) = s^{1-\mu}t^{\mu}x$. The left-hand side becomes $g(s^{1-\beta}(s^{1-\mu}t^{\mu})^{\beta}h(a, x), t^{1-\gamma}(s^{1-\mu}t^{\mu})^{\gamma}k(b, x))$ or

$$s^{[(1-\beta)(1-\alpha)+(1-\mu)\beta(1-\alpha)+(1-\mu)\alpha\gamma]}t^{[\mu\beta(1-\alpha)+(1-\gamma)\alpha+\mu\alpha\gamma]}g(h(a, x), k(b, x)).$$

From $g(h(a, x), k(b, x)) = x$, (3.3) and (3.4), it is equal to $s^{1-\mu}t^{\mu}x$. This completes the proof. $\square$

Remark 3.5. We note that the condition (3.1) holds and $\mu \in [0, 1)$ for $\alpha, \beta \in [0, 1)$ and $\gamma \in [0, 1]$. If $\alpha = \beta = \gamma$, then $\mu = \alpha$.

Corollary 3.6. *Let $g : \Omega^2 \to \Omega$ be a monotone and $\alpha(0 \leqslant \alpha < 1)$-jointly homogeneous map. Then for each $a \in \Omega$, the equation $x = g(a, x)$ has a unique solution in Ω. Moreover, the solution map over a is order preserving and homogeneous of degree* 1.

Proof. Let $a \in \Omega$. Define $h(x, y) = x$ and $k(x, y) = y$, the first and second projection maps, respectively. Then h and k are monotone and 0, 1-jointly homogeneous, respectively. Since $\alpha \neq 1$, we can apply the previous theorem. From $\mu = 0$, the solution map $\omega(a, b)$ is 0-jointly homogeneous. Since $x = g(a, x)$ if and only if $x = g(h(a, x), k(a, x))$, the solution map of $x = g(a, x)$ over a coincides with the map $a \mapsto \omega(a, a)$, which is a homogeneous map of degree 1. $\square$

Example 3.7. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$, and let $g : \Omega^2 \to \Omega$ defined by

$$g((x_1, x_2), (y_1, y_2)) = (\sqrt{x_1y_1} \wedge \sqrt{x_2y_2}, \sqrt{x_1y_2} \wedge \sqrt{x_2y_1}),$$

where $x \wedge y = \min\{x, y\}$. Then g is monotone and 1/2-jointly homogeneous and the solution map of $(x_1, x_2) = g(a, b, x_1, x_2)$ is given by $(a, b) \mapsto (a \wedge b, a \wedge b)$, which is order preserving and homogeneous of degree 1.

Corollary 3.8. *Let $g, h, k : \Omega^2 \to \Omega$ be monotone and α, β, γ-jointly homogeneous maps with $\alpha, \beta \in [0, 1)$ and $\gamma \in [0, 1]$. Let $\omega : \Omega^2 \to \Omega$ be the solution map of* (3.2). *Then for each $a \in \Omega$, the equation $x = \omega(a, x)$ has a unique solution in Ω and the solution map over a is order preserving and homogeneous of degree* 1.

Proof. This follows from Theorem 3.4, Remark 3.5 and Corollary 3.6. $\square$

Finally, we suggest a method constructing a sequence of monotone and α-jointly homogeneous maps.

Corollary 3.9. *Let $g : \Omega^2 \to \Omega$ be a monotone and α $(0 \leqslant \alpha < 1)$-jointly homogeneous map. Then there exists a sequence of monotone and α-jointly homogeneous maps g_n satisfying*

$$g_{n+1}(x, y) = g_n(g_n(x, g_{n+1}(x, y)), g_n(y, g_{n+1}(x, y))), \quad g_0 := g. \tag{3.5}$$

Proof. Setting $h = k = g$, we have a monotone and α-jointly homogeneous map g_1 such that $g_1(a, b) = g(g(a, g_1(a, b)), g(b, g_1(a, b)))$ for all $a, b \in \Omega$. Replacing g with g_1 yields a monotone and α-jointly homogeneous map g_2. Inductively, we obtain a monotone and α-jointly homogeneous map g_n. □

Remark 3.10. Starting with $g = g_0$ and g_1 in the previous theorem, we have several alternative processes contracting a sequence of monotone and α-jointly homogeneous maps. For examples

$$g_{n+1}(a, b) = g(g_n(a, g_{n+1}(a, b)), g_n(b, g_{n+1}(a, b))),$$
$$g_{n+1}(a, b) = g(g_{n-1}(a, g_{n+1}(a, b)), g_n(b, g_{n+1}(a, b)))$$

or $g_{n+2}(a, b) = g_{n-1}(g_n(a, g_{n+2}(a, b)), g_{n+1}(b, g_{n+2}(a, b))), \quad g_2 = g(g_2 = g_1)$.

4. Applications

In this section, we apply our results to the cone of positive definite operators on a Hilbert space. We recall the Löwner–Heinz inequality of positive definite operators, which says that if X and Y are positive definite operators with $X \leqslant Y$ then $X^\alpha \leqslant Y^\alpha$ for any $\alpha \in [0, 1]$ [14,7]. The two-variable Löwner–Heinz inequality holds true [1,12]: If $0 \leqslant X \leqslant X'$ and $0 \leqslant Y \leqslant Y'$, then

$$X\#_\alpha Y \leqslant X'\#_\alpha Y', \quad \alpha \in [0, 1].$$

This implies that the map $X \mapsto X^\alpha$, $\alpha \in [0, 1]$ is an order preserving homogeneous map of degree α, and the map $(X, Y) \mapsto X\#_\alpha Y$ is monotone and α-jointly homogeneous.

By Proposition 3.2, we have the following non-positive curvature property of the Thompson metric on the convex cone of positive definite operators.

Theorem 4.1 (Löwner-Heinz inequality and Non-positive curvature [3]). *For positive definite operators X, Y and $\alpha \in [0, 1]$*

$$d(X^\alpha, Y^\alpha) \leqslant \alpha d(X, Y).$$

Theorem 4.2. *Let A, B be positive definite operators on a Hilbert space $\mathscr{H}$, and let M, N be invertible operators on $\mathscr{H}$. Then the equation*

$$X = (A\#_\beta MXM^*)\#_\alpha(B\#_\gamma NXN^*), \quad \alpha, \beta \in [0, 1), \quad \gamma \in [0, 1] \tag{4.6}$$

has a unique positive definite solution and furthermore the solution map $\omega(A, B)$ varying over A and B is monotone and $\frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}$-jointly homogeneous.

Proof. It is straightforward to see that the map $(X, Y) \mapsto X\#_\alpha MYM^*$ is α-jointly homogeneous. The Löwner–Heinz inequality implies that the map $(X, Y) \mapsto X\#_\alpha MYM^*$ is monotone. Then the proof follows from Theorem 3.4. □

Example 4.3. We consider the case $M = N = I_{\mathscr{H}}$, the identity operator. Then the equation

$$X = (A\#_\beta X)\#_\alpha(B\#_\gamma X), \quad \alpha, \beta \in [0, 1), \quad \gamma \in [0, 1] \tag{4.7}$$

has the unique positive definite solution

$$X = A\#_\mu B, \quad \mu := \frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}.$$

If $\beta = \gamma$, then $X = A\#_\alpha B$.

Indeed, Eq. (4.7) is equivalent to

$$I = (X^{-1/2}AX^{-1/2}\#_\beta I)\#_\alpha(X^{-1/2}BX^{-1/2}\#_\gamma I). \tag{4.8}$$

If $\alpha = 0$, then $\mu = 0$ and $X = A$ satisfies (4.8). Suppose that $\alpha \neq 0$. From $A\#_\alpha I = A^{1-\alpha}$, and $A\#_\alpha B = I$ if and only if $B = A^{\frac{\alpha-1}{\alpha}}$, Eq. (4.8) is equivalent to $(X^{-1/2}BX^{-1/2})^{1-\gamma} = (X^{-1/2}AX^{-1/2})^{\frac{(1-\beta)(\alpha-1)}{\alpha}}$ or

$$B = X^{1/2}\left(X^{-1/2}AX^{-1/2}\right)^{\frac{(1-\beta)(\alpha-1)}{\alpha(1-\gamma)}}X^{1/2} = X\#_\tau A, \quad \tau := \frac{(1-\beta)(\alpha-1)}{\alpha(1-\gamma)}.$$

Since $A\#_t B = B\#_{1-t}A$, it becomes $B = A\#_{1-\tau}X$ and has the solution $X = A\#_{\frac{1}{1-\tau}}B$. One can see that $\frac{1}{1-\tau} = \frac{\alpha(1-\gamma)}{1-\beta(1-\alpha)-\alpha\gamma}$.

Example 4.4 (*Bushell equation*). Let $\alpha = 0$. Then Eq. (4.6) becomes $X = A\#_\beta MXM^* = A^{1/2}(A^{-1/2}(MXM^*)A^{-1/2})^\beta A^{1/2}$ and has a unique positive definite solution. In particular if $A = I$ and $\beta = 1/n$, then the equation $X = (MXM^*)^{1/n}$ appears in [5,6]. See [13] for a connection between (solution maps of) Bushell equations and polar decompositions of M.

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